An Improved Partitioned Element Method for Constructing Higher-Order Shape Functions on Arbitrary Polyhedra

Brian Giffin, Mark Rashid

Department of Civil and Environmental Engineering

University of California, Davis

Polygonal and Polyhedral Discretizations in Computational Mechanics

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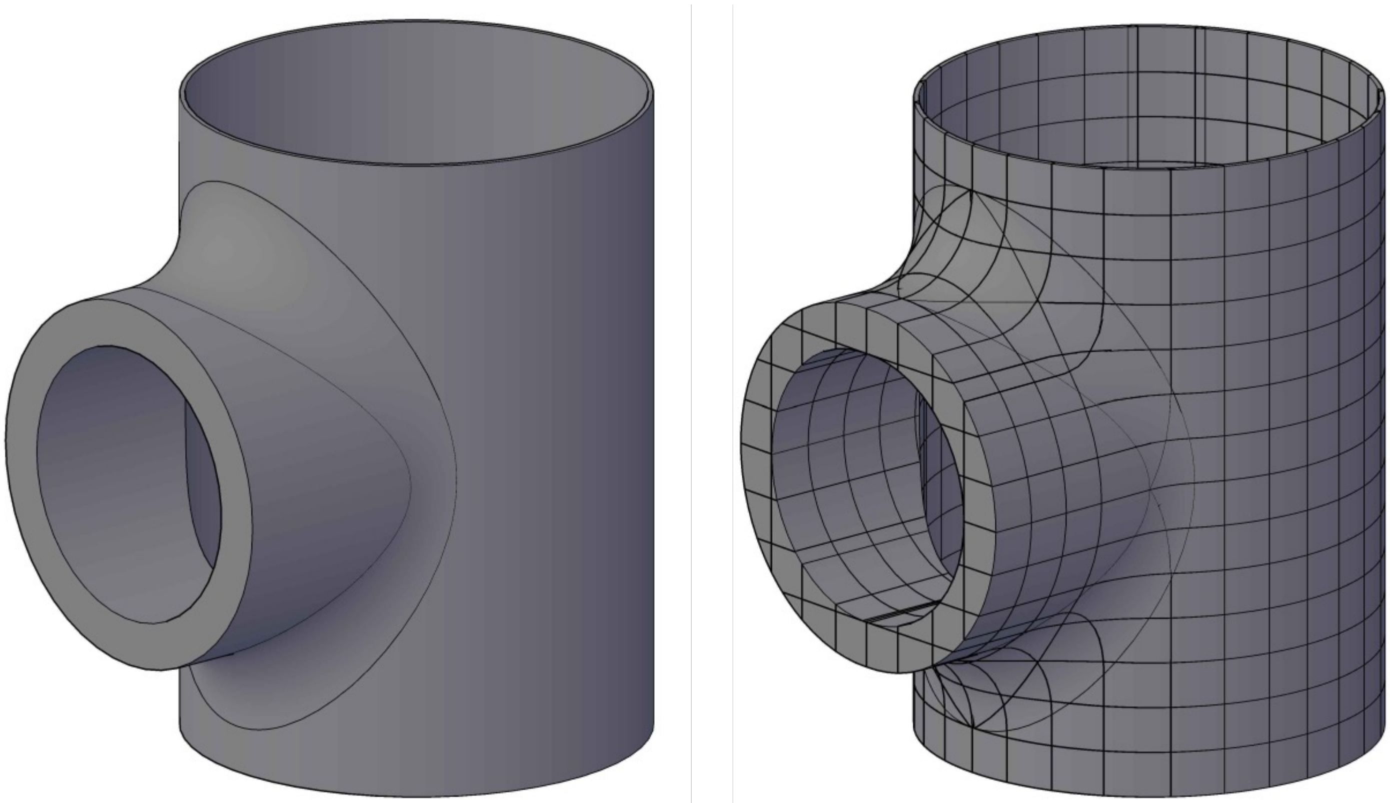
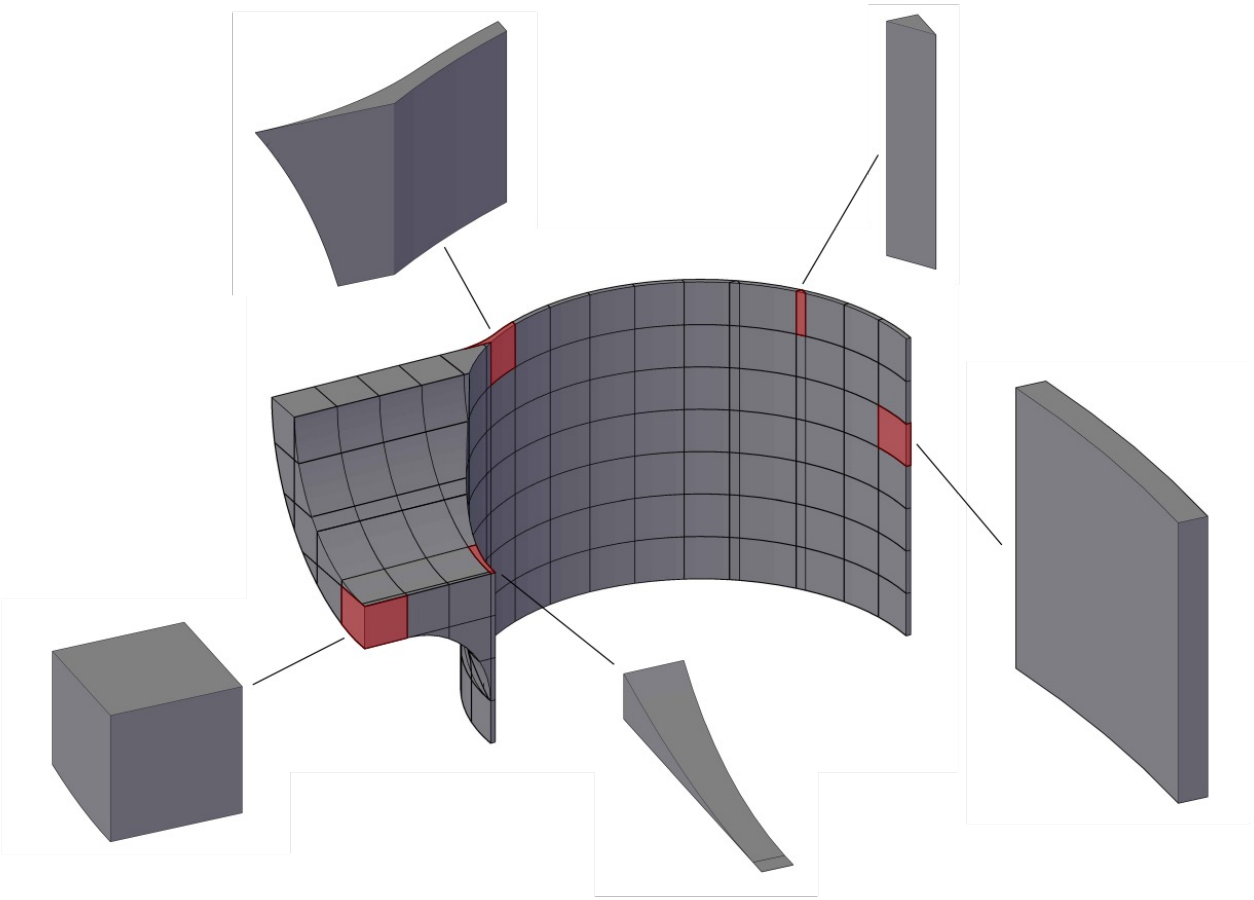
Motivation

1) Automate the meshing process…



Motivation

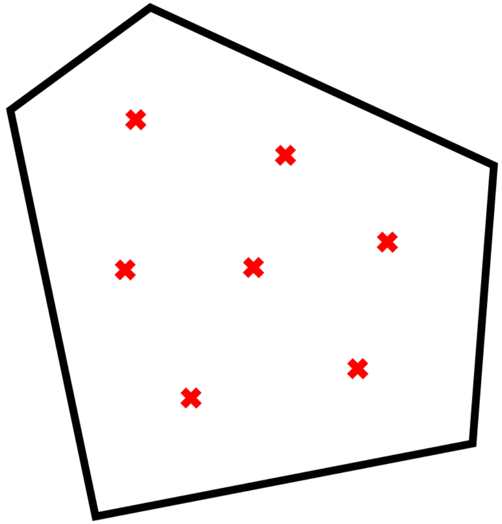
2) …and improve element quality/robustness



Solid mechanics applications

$$\int_{\Omega} \sigma_{ij} v_{i,j}\, d\Omega + \int_{\Omega} \rho(a_i - \bar{b}_i) v_i\, d\Omega - \int_{\Gamma} \bar{t}_i v_i\, d\Gamma = 0$$

- Maintain material state data for a finite collection of material points
  - Must define a discrete quadrature rule

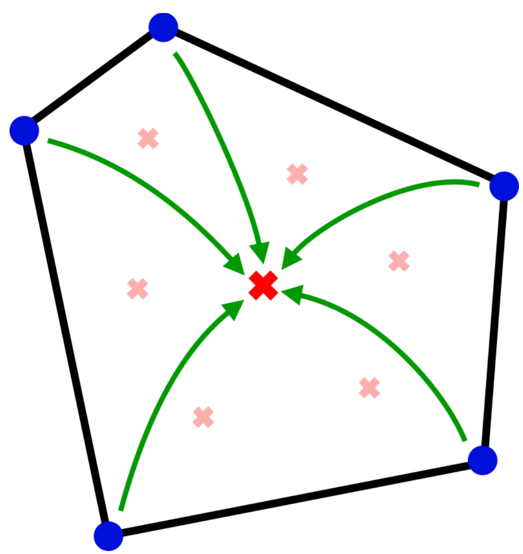


Discrete interpolation operators

$$u(\mathbf{x}) = \sum_a \varphi_a(\mathbf{x})\, u_a \qquad \nabla u(\mathbf{x}) = \sum_a \nabla \varphi_a(\mathbf{x})\, u_a$$

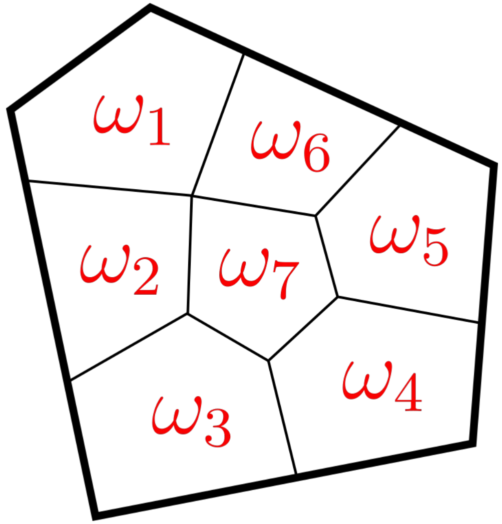
$$u_q = \sum_a \varphi_a^q\, u_a \qquad \nabla u_q = \sum_a \nabla \varphi_a^q\, u_a$$

Ultimately, we need linear mappings from nodal values to interpolated values and gradients at quadrature points



Partitioned element method

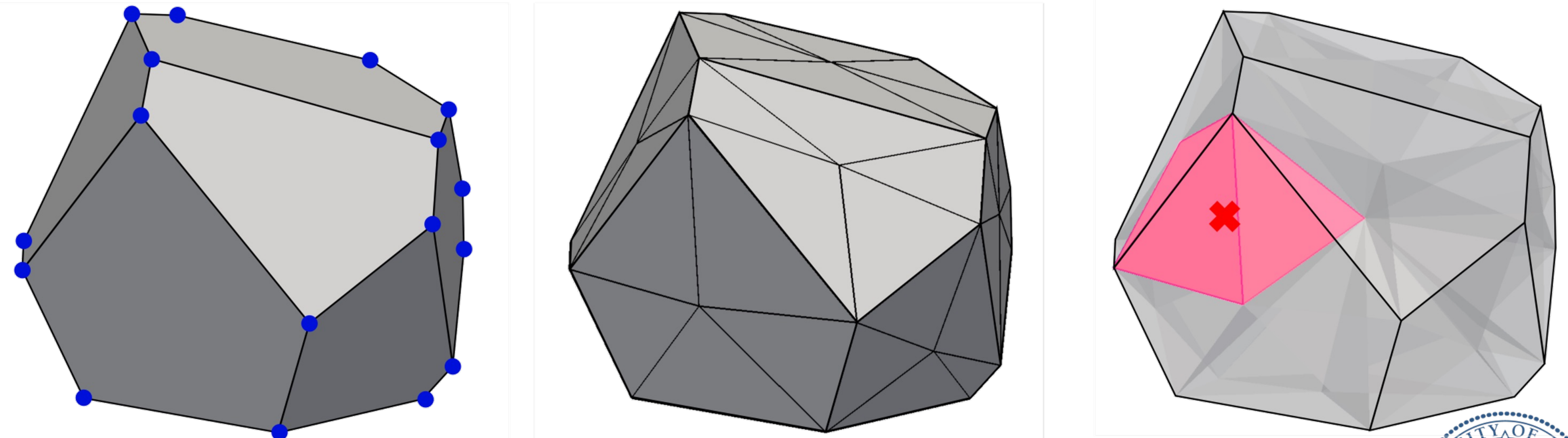
- Partition elements into cells: $\omega_j \subset \Omega_e$
- Solve local approximation problems on the partitioned geometry
  - Obtain linear mappings that *approximate* **smooth**, **continuous** functions



Element partition

Element is partitioned into quadrature "cells"

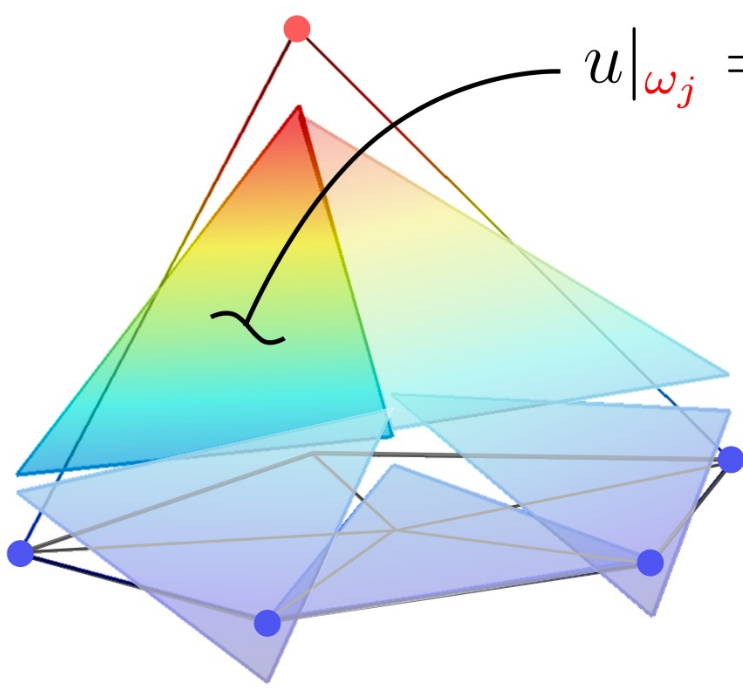
- Quadrature weights equal to cell volumes
- Values and gradients evaluated at cell centroids



PEM approximation space

Solution representation is piece-wise polynomial in each cell (discontinuous at cell boundaries)

$$\mathcal{P}_k(\Omega_e) = \{u : u|_{\omega_j} \in P_k(\omega_j)\ \forall \omega_j \subset \Omega_e\}$$

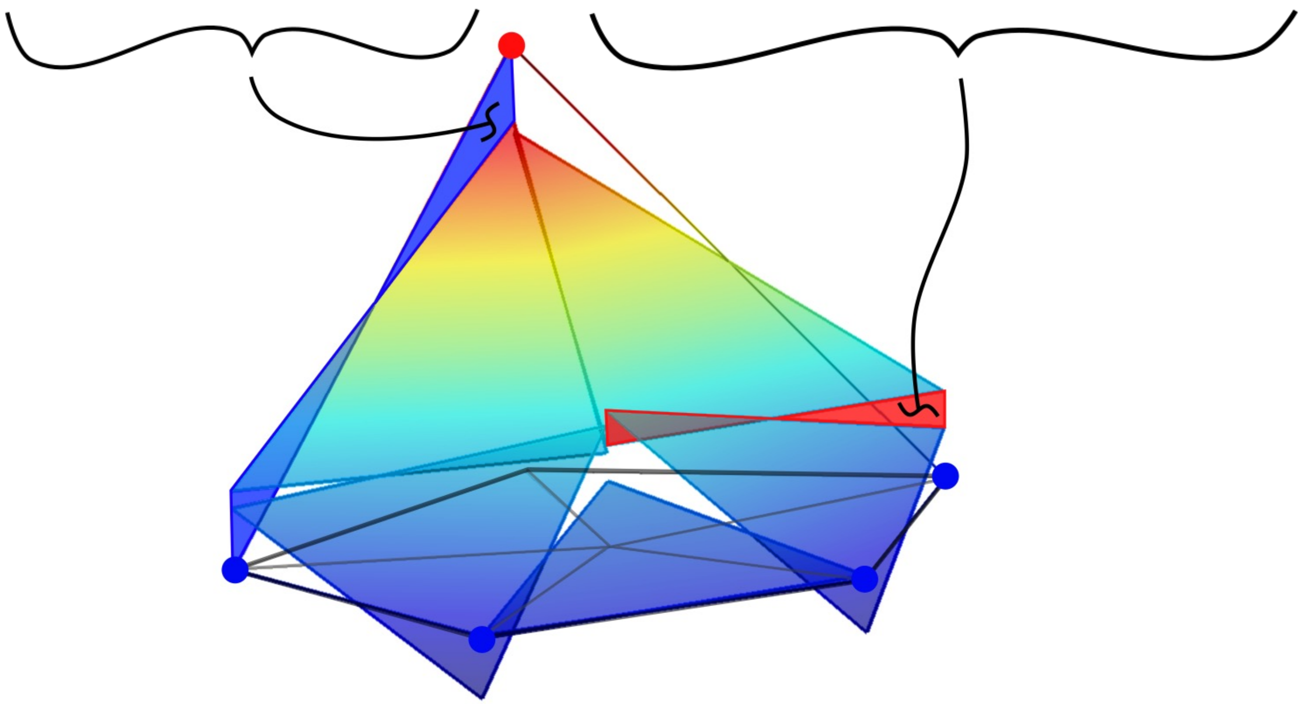


$$u|_{\omega_j} = c + g_x x + g_y y$$

Weighted minimization of interface discontinuities

$$\min_u \Psi(u)$$

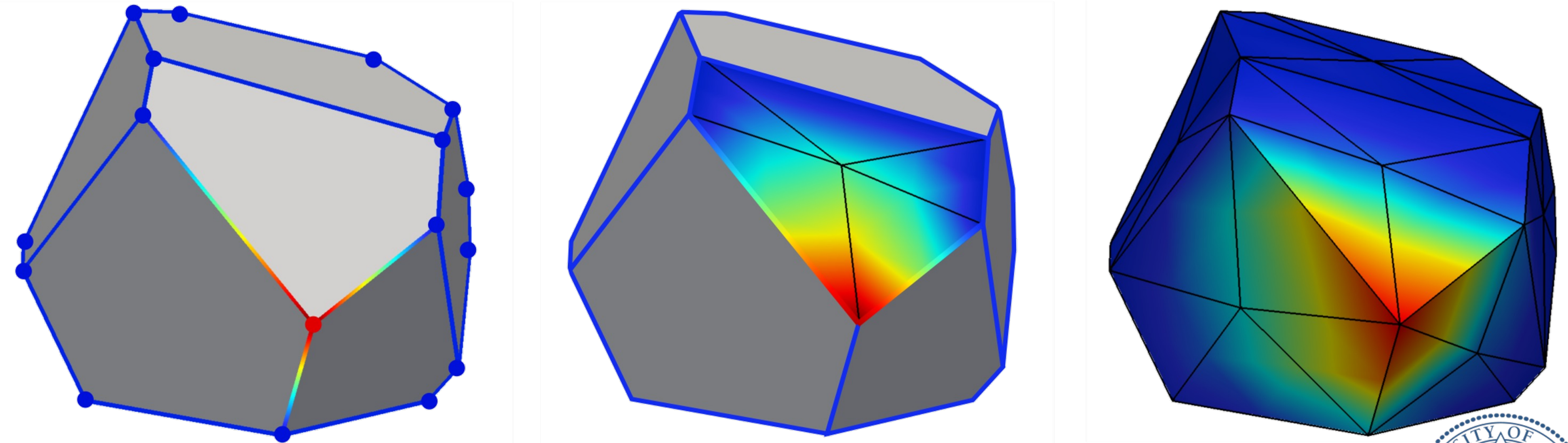
$$\Psi(u) \equiv \frac{1}{2} \sum_{j \in \mathcal{F}_b} \beta_b \int_{F_j} [\![u]\!]^2 \, ds + \frac{1}{2} \sum_{|\alpha|=0}^{k} \sum_{j \in \mathcal{F}_0} \frac{\beta_{|\alpha|}}{h^{2|\alpha|}} \int_{F_j} [\![D^\alpha u]\!]^2 \, ds$$



Hierarchical construction of shape functions

Approximants are constructed in sequence:

Nodes ⇨ Edges ⇨ Faces ⇨ Element



Constraints

Continuity:

$$\int_F [\![u]\!] \, da = 0 \qquad \forall F \subset \partial\Omega_e$$

Reproducibility:

$$x_q^\alpha = \sum_a \varphi_a^q \, x_a^\alpha \qquad \nabla x_q^\alpha = \sum_a \nabla\varphi_a^q \, x_a^\alpha \quad \forall\, |\alpha| \leq k$$

Consistency:

$$\int_{\Omega_e} \nabla(x^\alpha \varphi) \, dv = \int_{\partial\Omega_e} x^\alpha \varphi \, \mathbf{n} \, da \quad \forall \varphi, \, |\alpha| \leq k - 1$$

Stability:

$$\inf_{u \in \mathbb{U}} \sup_{v \in \mathbb{V}} \frac{a(u, v)}{||u|| \, ||v||} > 0$$

Constraints

Continuity: enforced weakly via penalty terms

$$\int_F [\![u]\!] \, da = 0 \qquad \forall F \subset \partial\Omega_e$$

Reproducibility: obtained through minimization

$$x_q^\alpha = \sum_a \varphi_a^q \, x_a^\alpha \qquad \nabla x_q^\alpha = \sum_a \nabla \varphi_a^q \, x_a^\alpha \quad \forall \, |\alpha| \leq k$$

Consistency: enforced via Lagrange multipliers

$$\int_{\Omega_e} \nabla(x^\alpha \varphi) \, dv = \int_{\partial\Omega_e} x^\alpha \varphi \, \mathbf{n} \, da \quad \forall \varphi, \; |\alpha| \leq k - 1$$

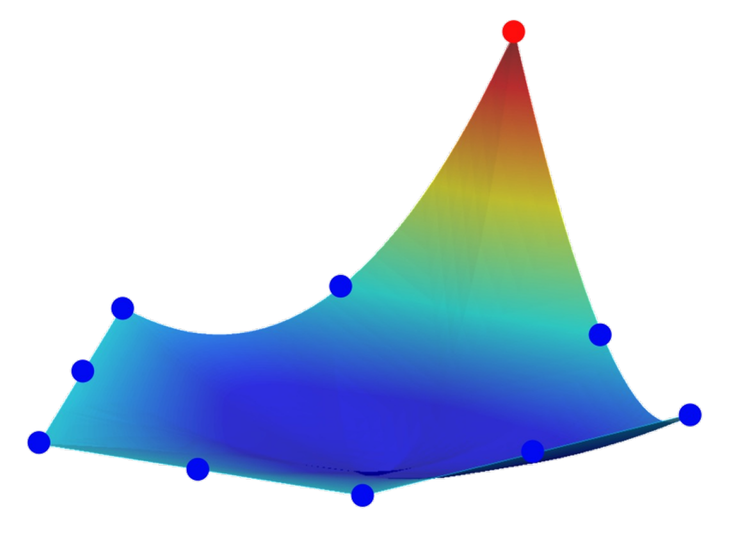
Stability: require sufficient quadrature cells

$$\inf_{u \in \mathbb{U}} \sup_{v \in \mathbb{V}} \frac{a\big(u, v\big)}{||u|| \, ||v||} > 0$$

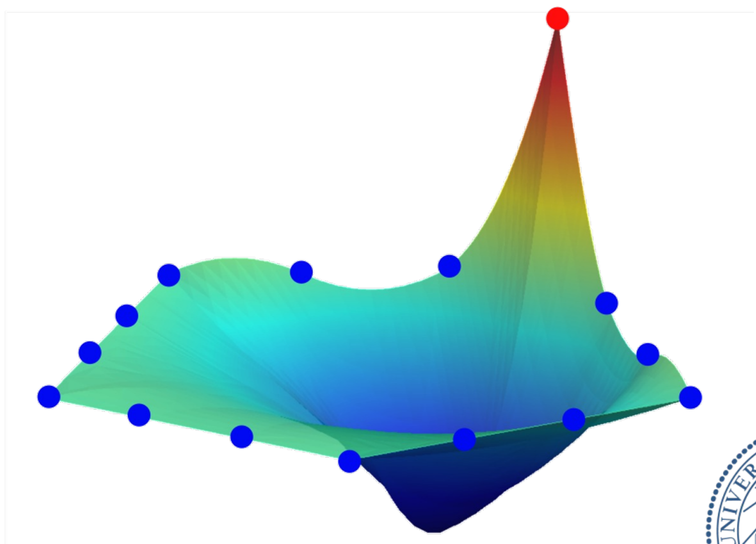
High-order PEM approximants

Requirements:

- Approximation space with high-order completeness
- High-order consistency constraint enforcement
- Sufficiently accurate quadrature rule



$k = 2$



$k = 3$

Over-constrained approximants

- Enforcing high-order consistency constraints on the shape functions over-constrains the PEM minimization problem
  - Results in loss of reproducibility

(Table of errors)

Petrov-Galerkin approach

Only enforce consistency on the **test functions**

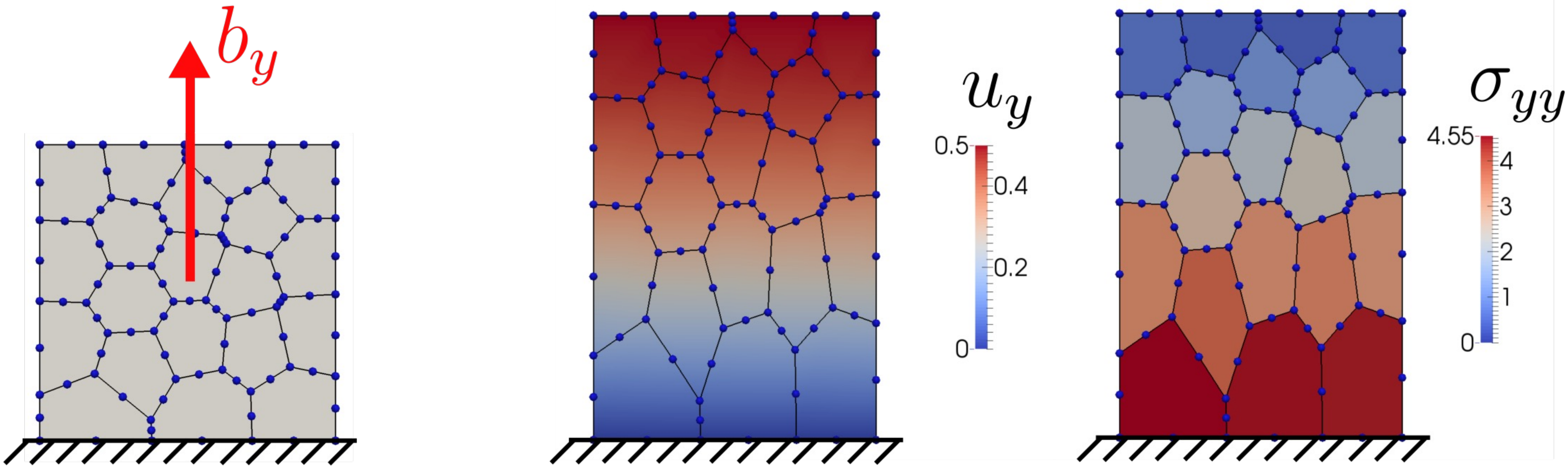
- Results in a Petrov-Galerkin scheme
  - Trial solution space maintains reproducibility
  - Can use low-order quadrature rules

Stability

- Test functions are obtained via L_2 projection of corresponding trial functions onto a constrained sub-space of the PEM approximation space
- Trial and test spaces differ only minimally
  - Well-balanced spaces
  - Sufficiently stable

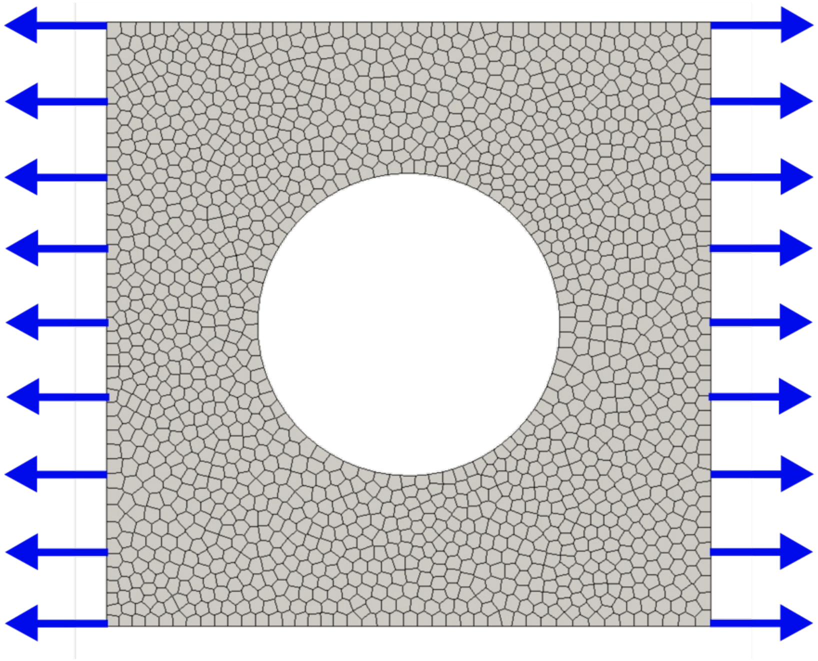
2nd order patch test

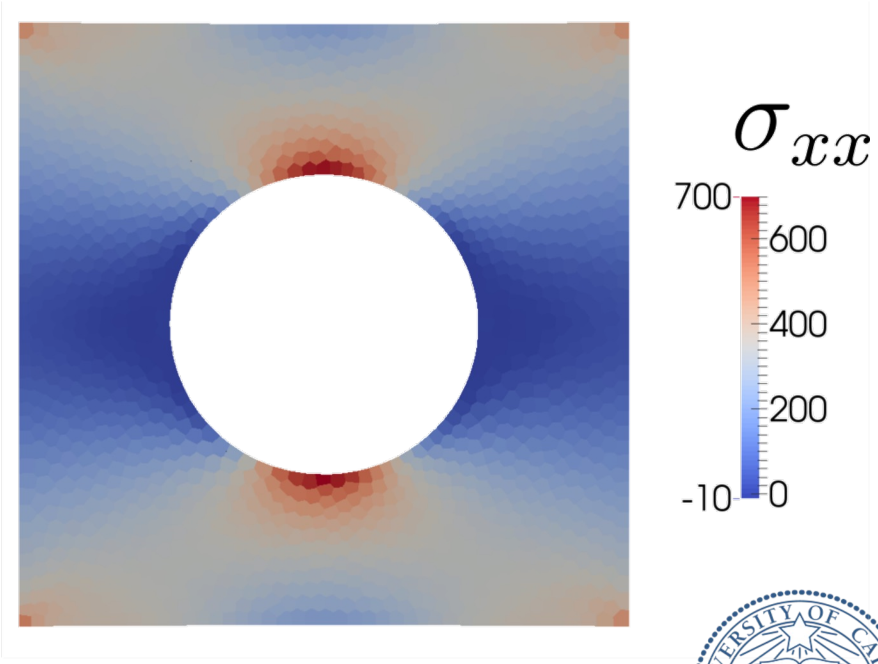
Employing the P-G approach, quadratic elements pass 2nd order patch tests



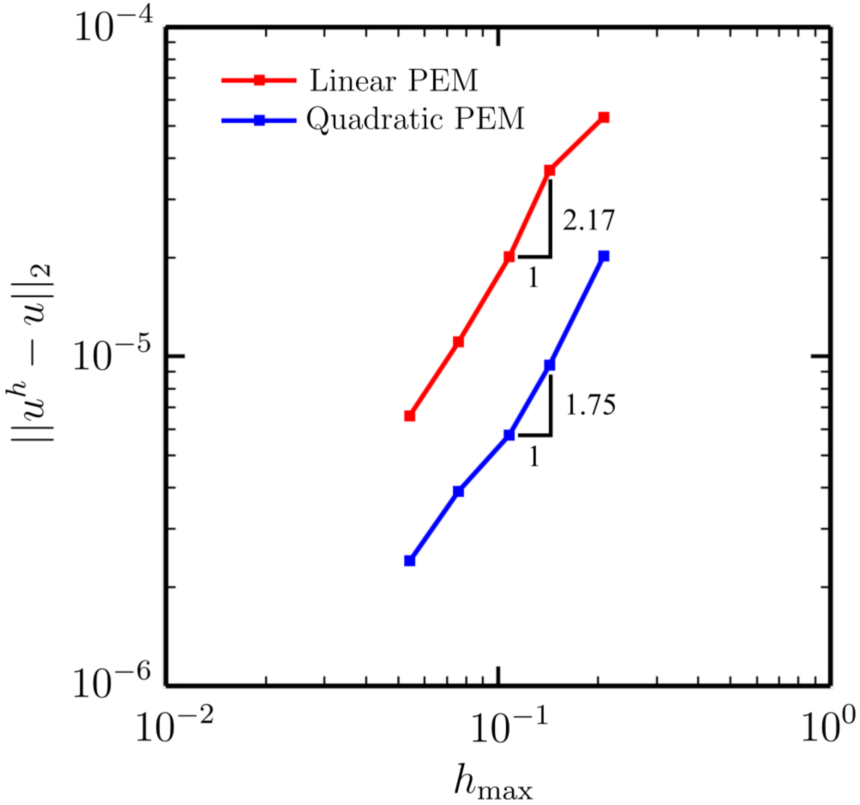
2d example problem

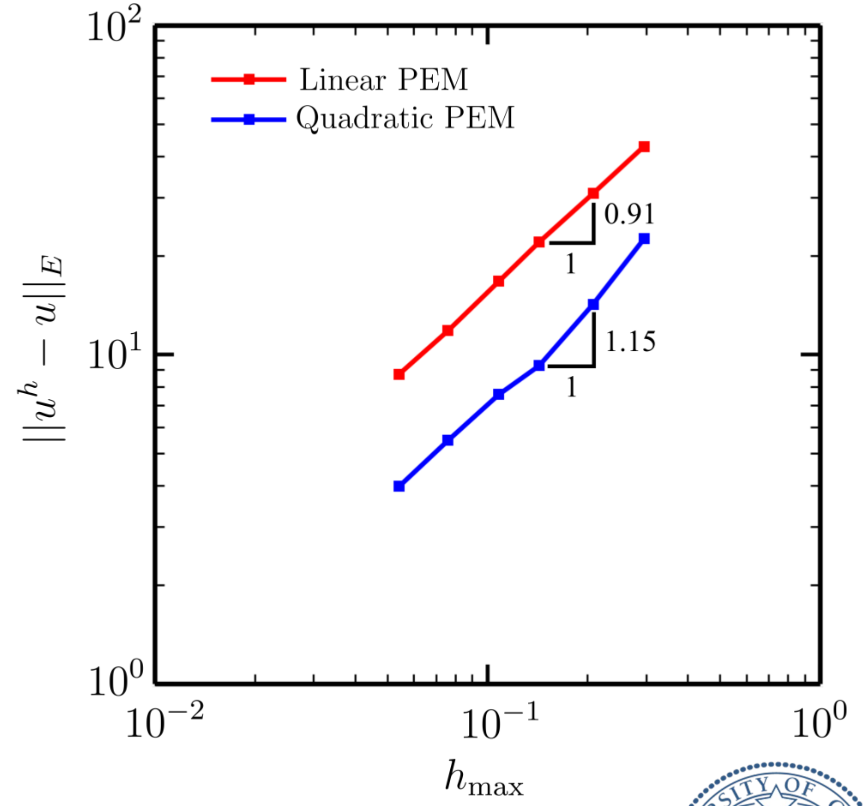
Examine solution accuracy and convergence for linear and quadratic polygonal PEM elements





Accuracy comparison





Ongoing efforts

- Try to recover optimal convergence rates
- Add internal element degrees of freedom
- Use selective p-refinement for thin domains

Questions?

bdgiffin@ucdavis.edu