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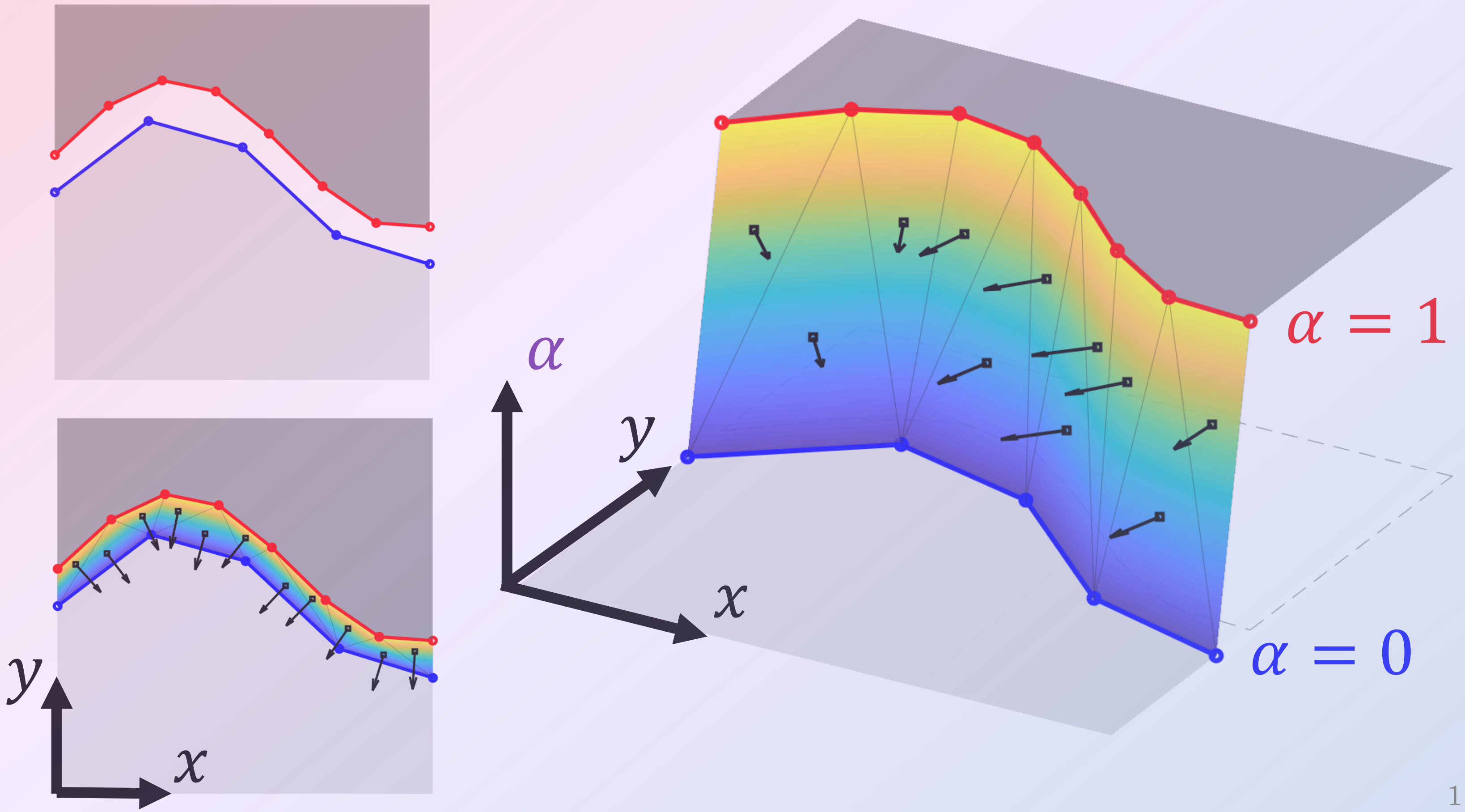
Contact and Interface Mechanics: Modeling and Computation

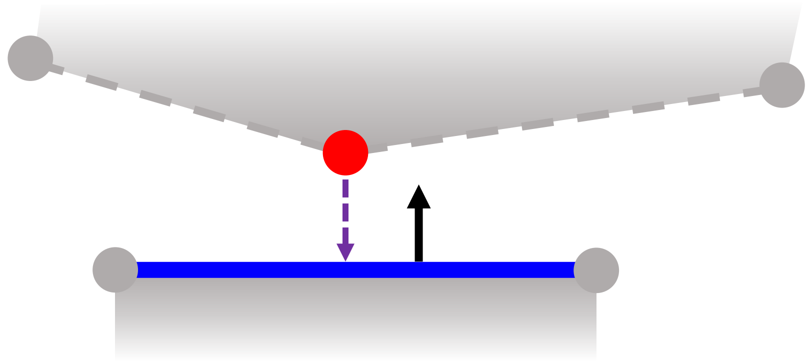
# Hyper-dimensional Gap Finite Elements for the Enforcement of Interfacial Constraints



# Outline

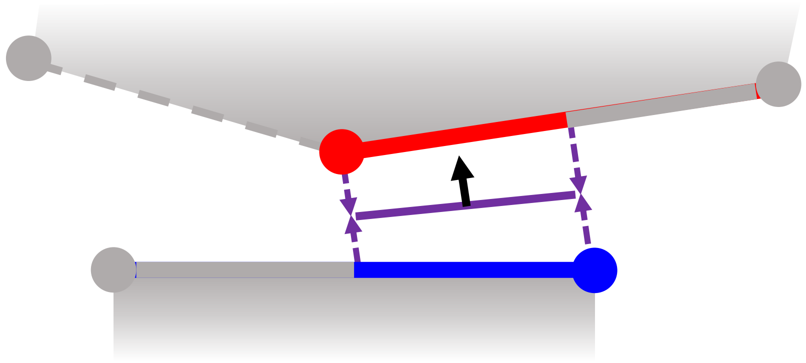
1. Limitations of conventional contact algorithms
2. The proposed contact discretization methodology
3. Distinguishing novelties and potential advantages
4. Demonstration of efficacy and preliminary results
5. Conclusions and future work

# Conventional contact enforcement methods possess several critical limitations



**Node-to-surface methods:**

- Non-symmetric behavior
- Non-smooth sliding (chatter)
- Poor solution accuracy (locking)



**Surface-to-surface methods:**

- Computationally expensive
- Difficult to implement
- Not easily generalized

# Mathematical statement of contact problems necessitates the definition of well-defined shared interfaces/boundaries between continuum bodies

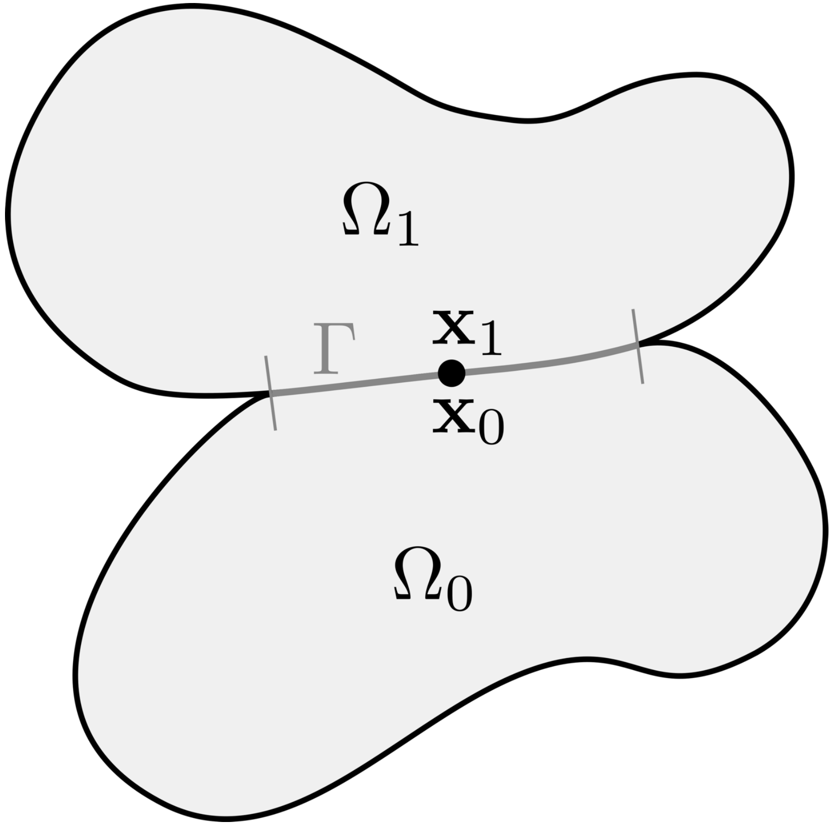
## Model problem

"Tied" interface constraint:

$$\boldsymbol{x}_1 - \boldsymbol{x}_0 = \mathbf{0} \quad \forall \boldsymbol{x} \in \Gamma$$

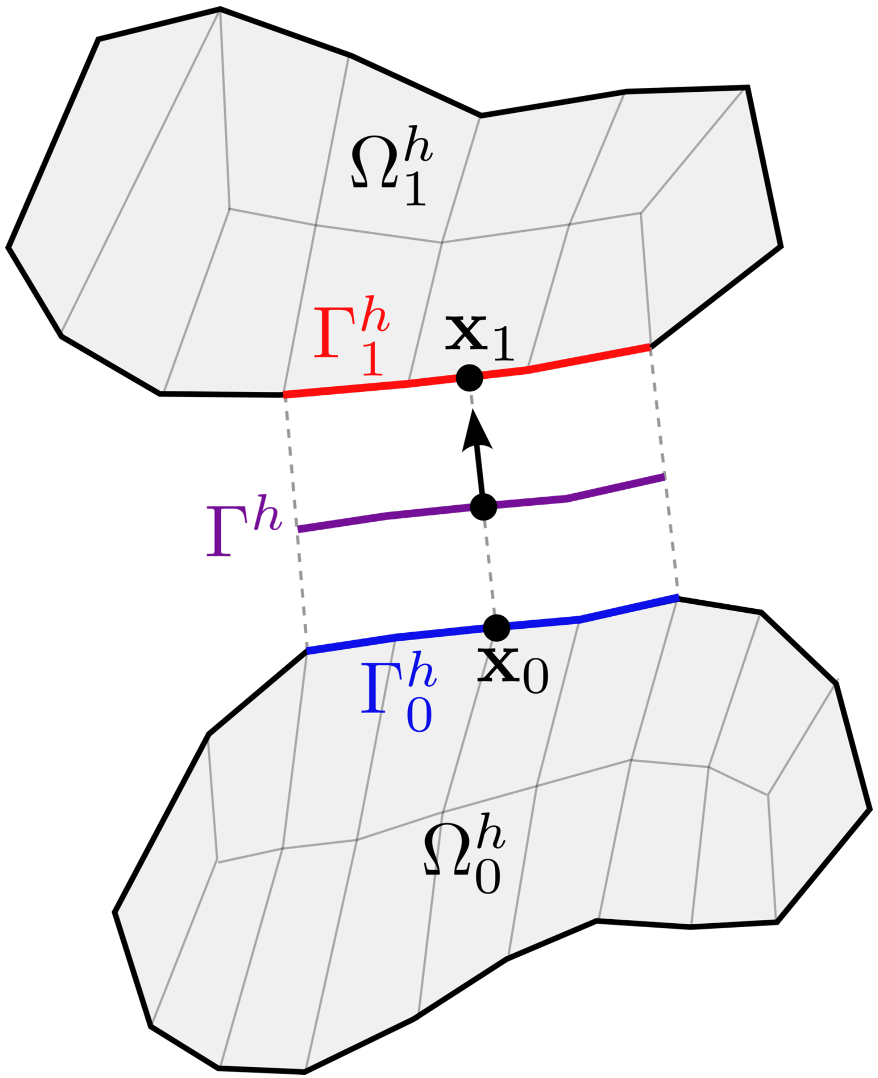
Enforcement using Lagrange multipliers:

$$\int_\Gamma (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma = 0 \quad \forall \boldsymbol{\lambda}$$



# Shared interface is ambiguously defined in a finite element discretization of the original BVP

- A discrete intermediate surface is defined
  - Chosen somewhat **arbitrarily**
- Point pairs on both bodies are related through a projective mapping
  - Computationally expensive
  - Mapping is not always unique or robust




$$\int_{\Gamma^h} (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma^h = 0 \quad \forall \boldsymbol{\lambda}$$

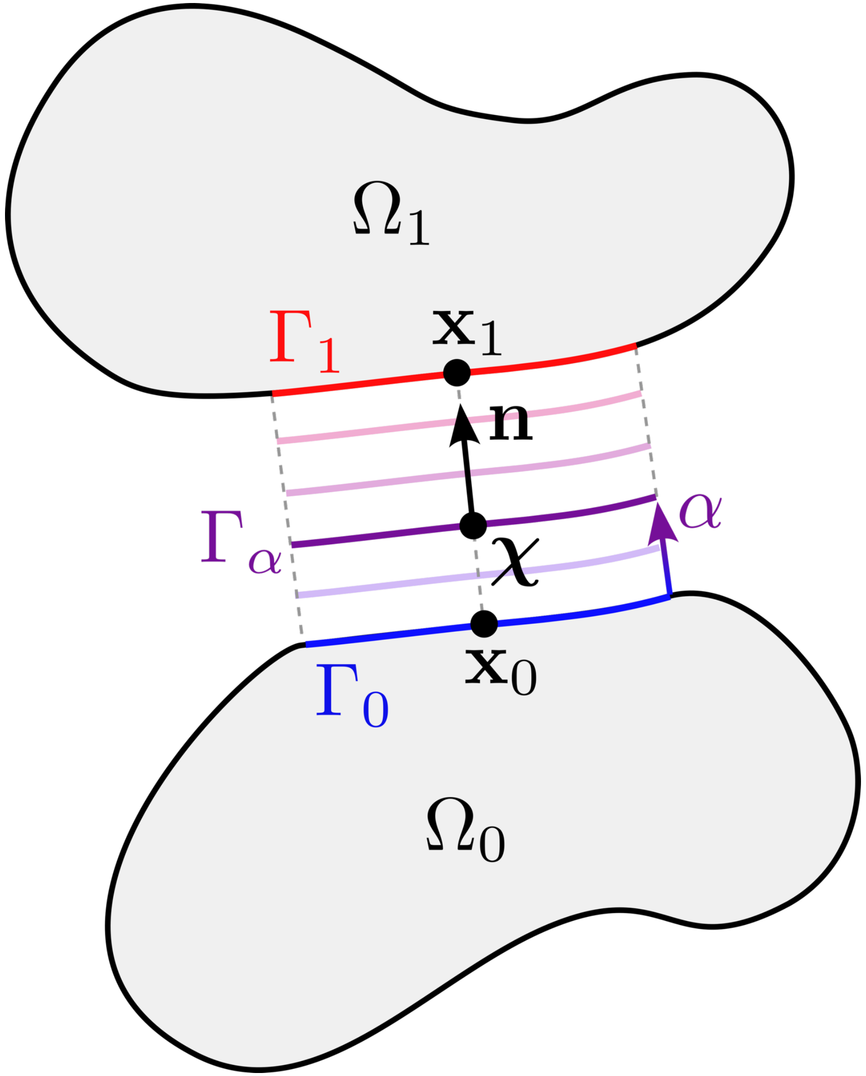
# Define a continuous family of intermediate surfaces parameterized by $\alpha \in [0,1]$

- Regard $\alpha$ as an auxiliary spatial coordinate

$$\boldsymbol{\chi} = (\boldsymbol{x}, \alpha)$$

- Mortar constraint integrals may be evaluated over a specific intermediate surface $\Gamma_\alpha$

$$\int_{\Gamma_\alpha} (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma_\alpha = 0 \quad \forall \boldsymbol{\lambda}$$

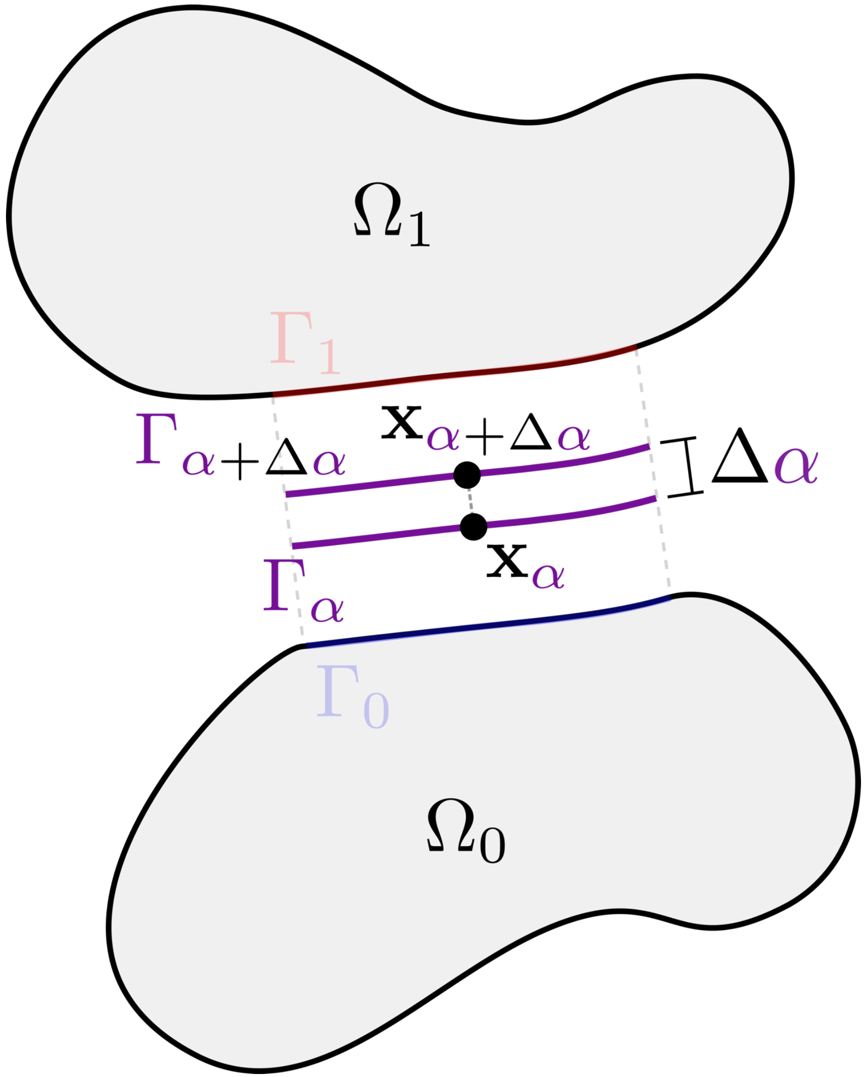


# Define a local measure of separation between adjacent intermediate surfaces parameterized by $\alpha$

$$(\boldsymbol{x}_1 - \boldsymbol{x}_0) = \mathbf{0} \quad \forall \boldsymbol{x} \in \Gamma$$

$$\boxed{\frac{\partial \boldsymbol{x}}{\partial \alpha} = \mathbf{0} \quad \forall \boldsymbol{x} \in \Gamma_\alpha}$$

$$\frac{\partial \boldsymbol{x}}{\partial \alpha} = \lim_{\Delta\alpha \to 0} \frac{\boldsymbol{x}_{\alpha+\Delta\alpha} - \boldsymbol{x}_\alpha}{\Delta\alpha}$$



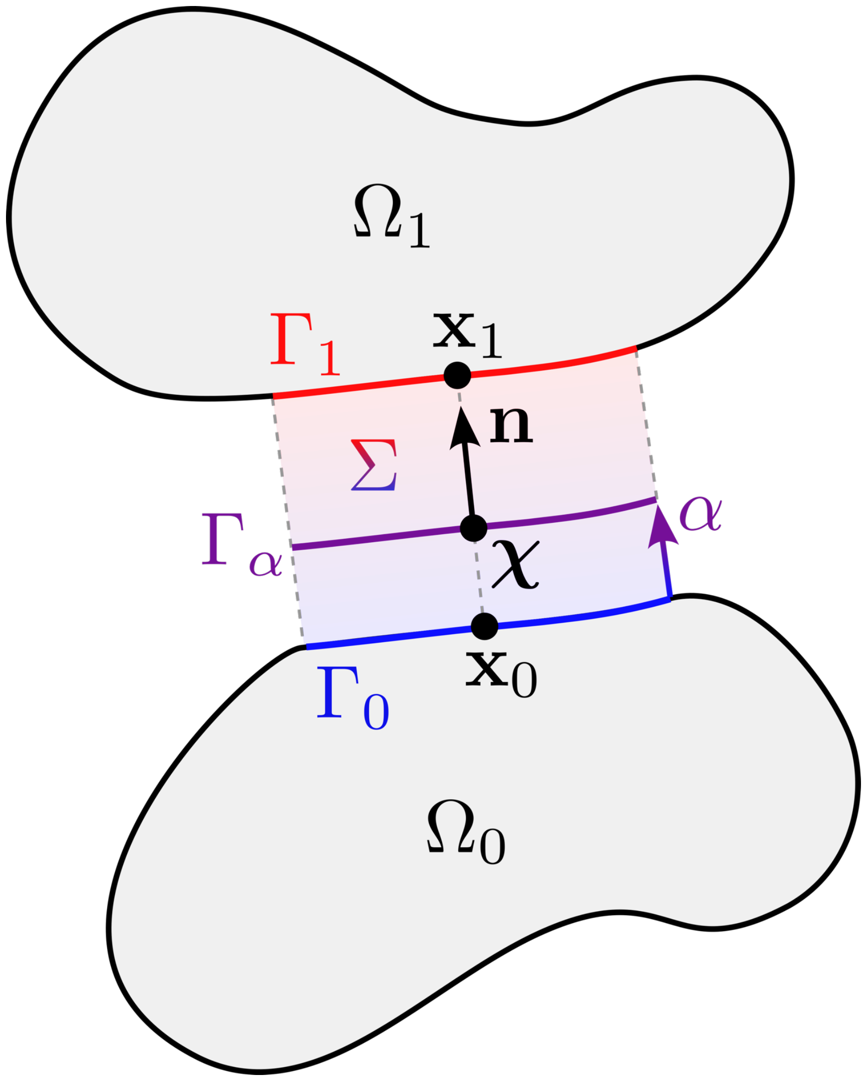
# Pose mortar integrals over the intermediate domain $\Sigma$ comprising all intermediate surfaces $\Gamma_\alpha$

$$\int_{\Gamma} (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma = 0 \quad \forall \boldsymbol{\lambda}$$

$$\boxed{\int_{\alpha=0}^{\alpha=1} \int_{\Gamma_\alpha} \frac{\partial \boldsymbol{x}}{\partial \alpha} \cdot \boldsymbol{\lambda} \, \mathrm{d}\Gamma_\alpha \, \mathrm{d}\alpha = 0 \quad \forall \boldsymbol{\lambda}}$$

- Regard $\Sigma$ as a differentiable manifold

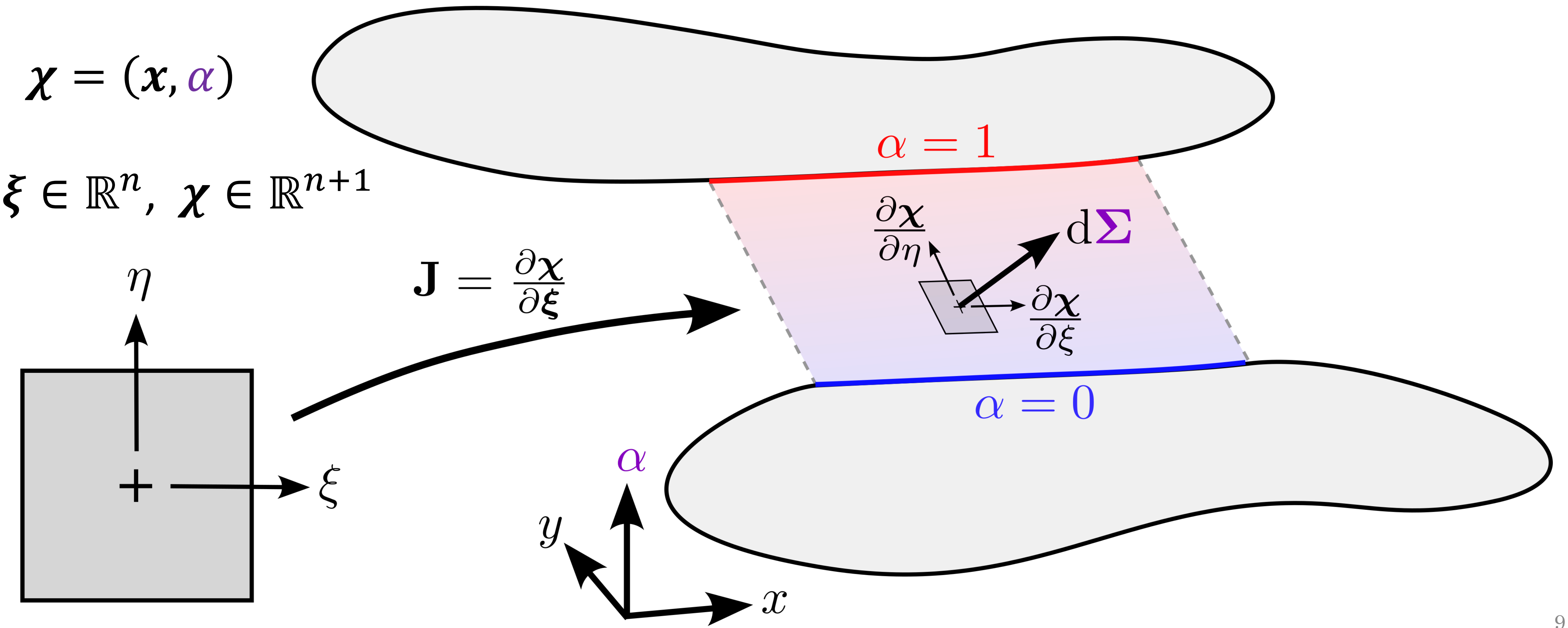
$$\Sigma = \{\Gamma_\alpha \mid \forall \alpha \in [0,1]\}$$



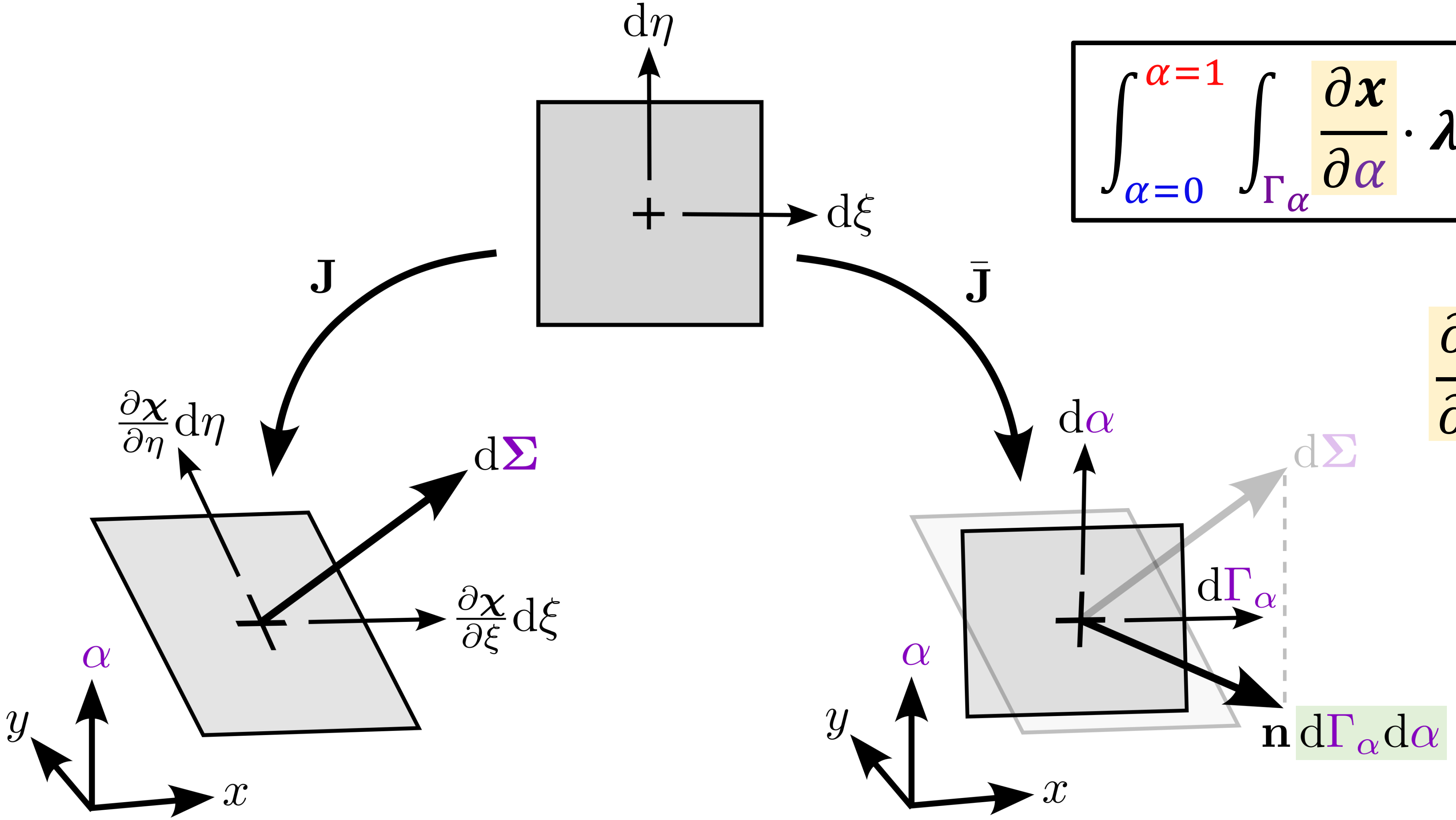
# Represent $\Sigma$ parametrically as an $n$-dimensional hyper-surface embedded in $\mathbb{R}^{n+1}$

$$\boldsymbol{\chi} = (\boldsymbol{x}, \alpha)$$

$$\boldsymbol{x}, \boldsymbol{\xi} \in \mathbb{R}^n, \ \boldsymbol{\chi} \in \mathbb{R}^{n+1}$$



# Projection of the directed hyper-surface area d$\Sigma$ onto the original spatial domain yields the desired differential form for evaluating mortar integrals




$$\boxed{\int_{\alpha=0}^{\alpha=1}\int_{\Gamma_\alpha}\frac{\partial \boldsymbol{x}}{\partial \alpha}\cdot\boldsymbol{\lambda}\,\mathrm{d}\Gamma_\alpha\,\mathrm{d}\alpha = 0 \quad \forall \boldsymbol{\lambda}}$$

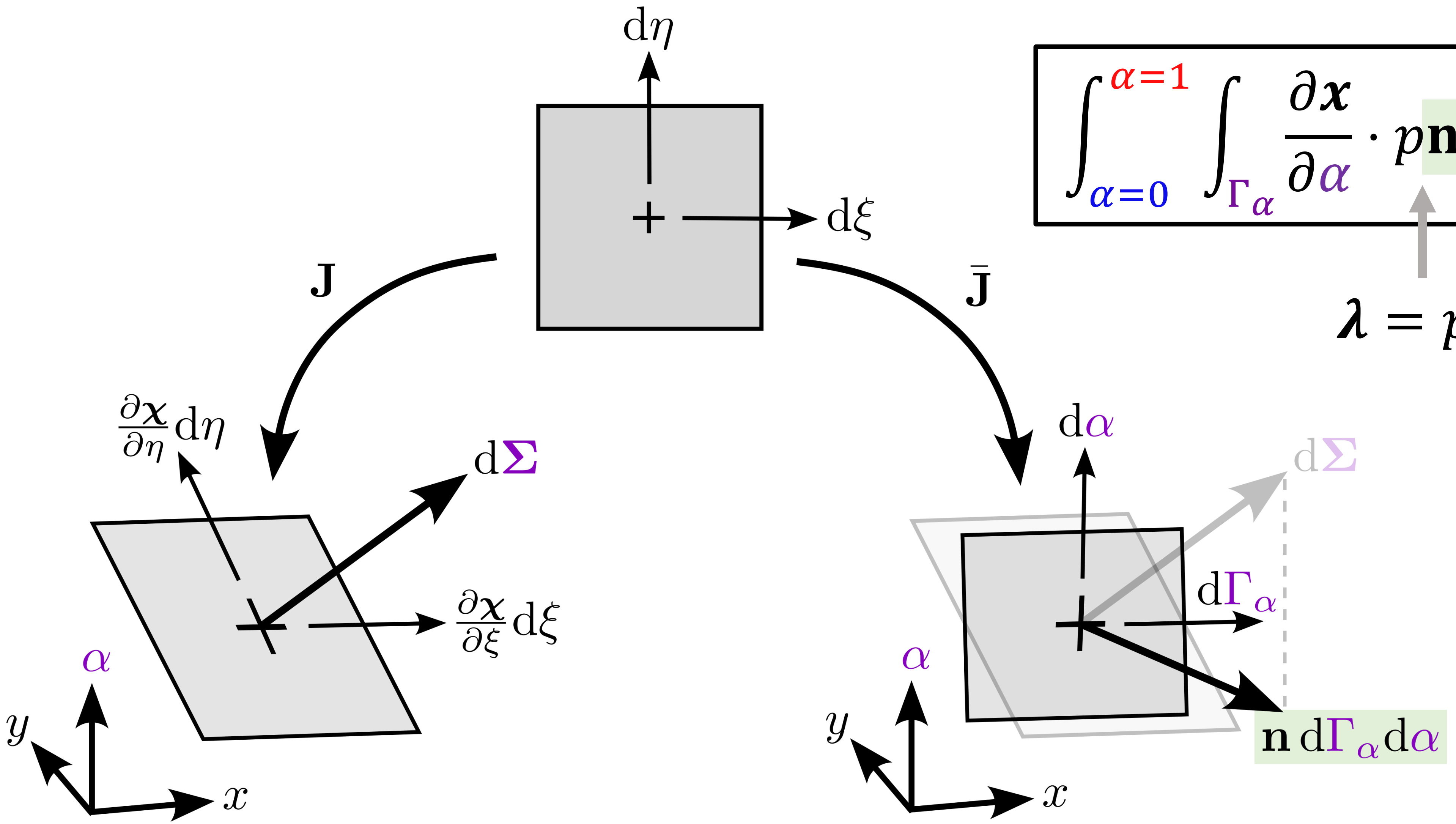
$$\frac{\partial \boldsymbol{x}}{\partial \alpha} = \frac{\partial \boldsymbol{x}}{\partial \xi}\,\bar{\boldsymbol{g}}^{-1}\,\frac{\partial \alpha}{\partial \xi}$$

$$\bar{\boldsymbol{g}} = \bar{\mathbf{J}}^T\bar{\mathbf{J}}$$

# The intermediate surface normal $\mathbf{n}$ may further be used to enforce normal contact constraints

$$\boxed{\int_{\alpha=0}^{\alpha=1} \int_{\Gamma_\alpha} \frac{\partial \boldsymbol{x}}{\partial \alpha} \cdot p\mathbf{n}\, \mathrm{d}\Gamma_\alpha\, \mathrm{d}\alpha = 0 \quad \forall p}$$

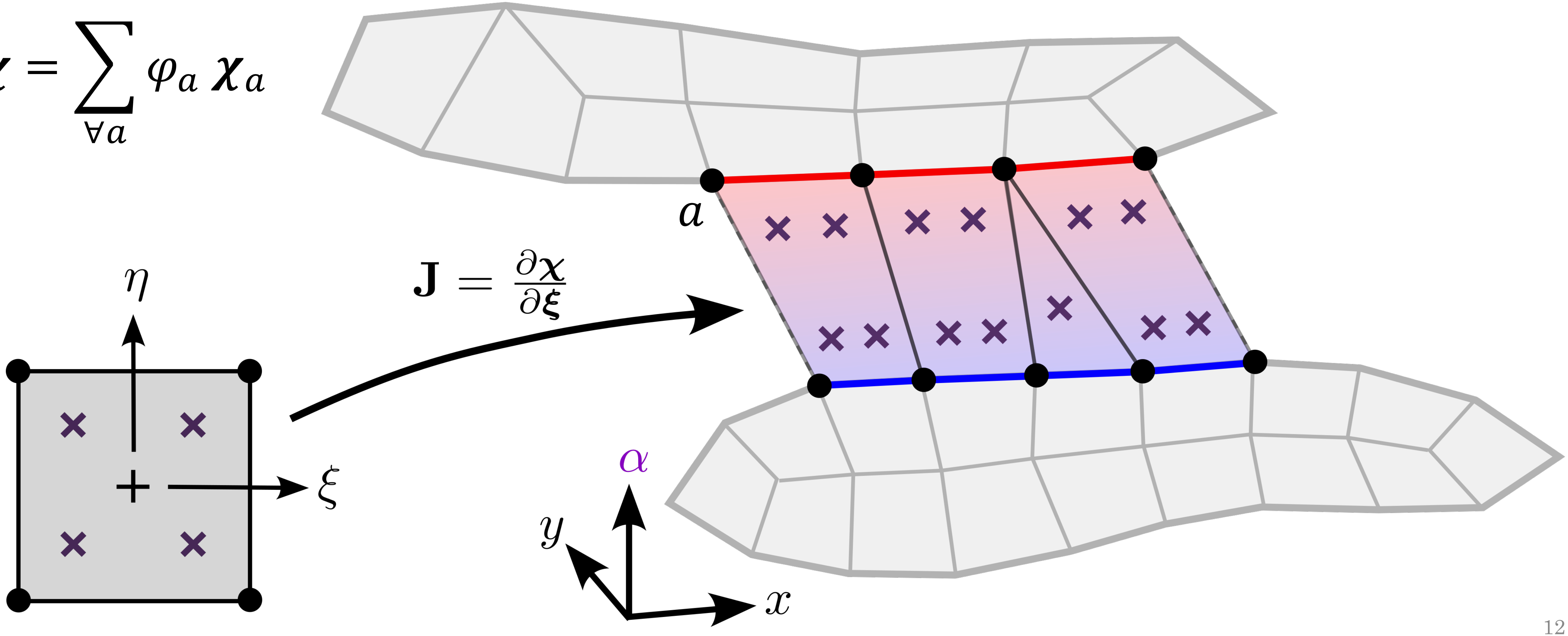
$$\boldsymbol{\lambda} = p\mathbf{n}$$



# Exploit a conformal discretization of the intermediate domain into finite elements

FEM basis:

$$\boldsymbol{\chi} = \sum_{\forall a} \varphi_a \, \boldsymbol{\chi}_a$$

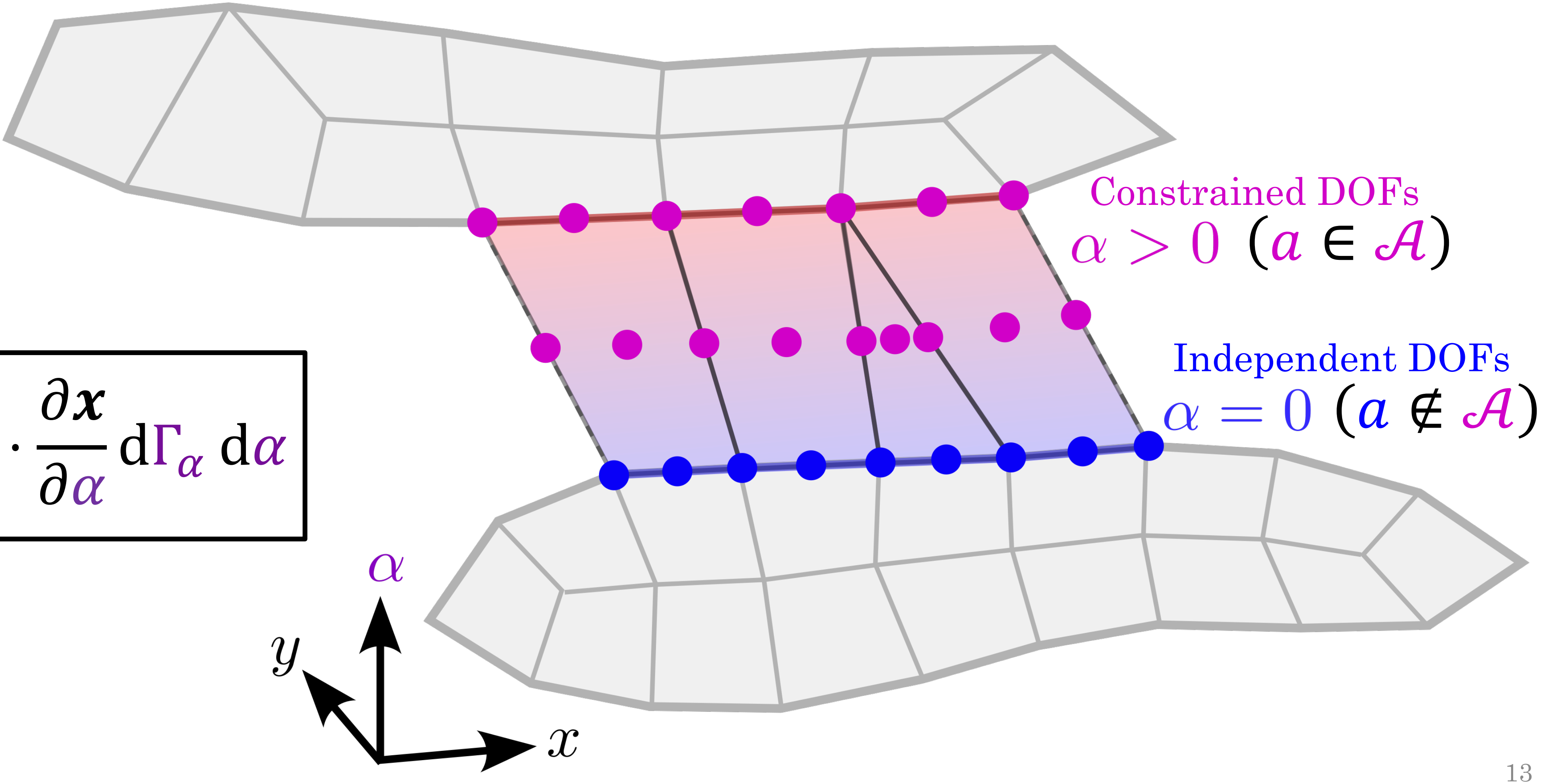


# Select the Lagrange multiplier basis consistent with $L^2$ minimization of $\frac{\partial x}{\partial \alpha}$ over the intermediate domain

Multiplier basis:

$$\boldsymbol{\lambda} = \sum_{\forall a \in \mathcal{A}} \frac{\partial \varphi_a}{\partial \alpha} \boldsymbol{\lambda}_a$$

$$\min_{\forall \boldsymbol{x} \notin \Gamma_0} \int_{\alpha=0}^{\alpha=1} \int_{\Gamma_\alpha} \frac{\partial \boldsymbol{x}}{\partial \alpha} \cdot \frac{\partial \boldsymbol{x}}{\partial \alpha} \, \mathrm{d}\Gamma_\alpha \, \mathrm{d}\alpha$$



# The proposed approach differs from other related volume-based interface discretization methods

1. The contact domain method (Oliver et al., 2009)
2. The third medium approach (Wriggers et al., 2013)
3. Contact layer elements (Weißenfels and Wriggers, 2015)
4. Fictitious contact material method (Bog et al., 2015)

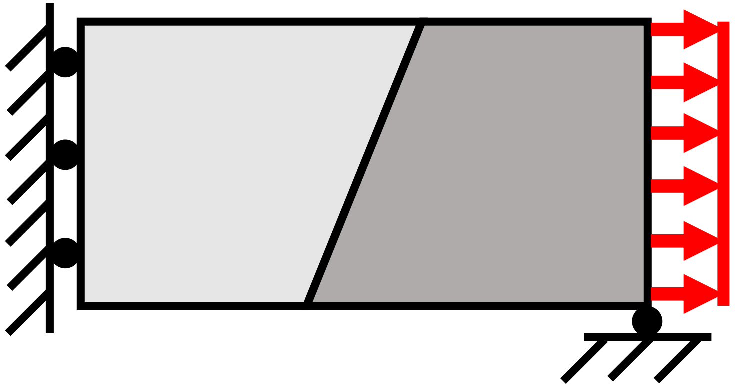
Introduction of the hyper-dimensional coordinate $\alpha$ constitutes a distinguishing novelty of the method

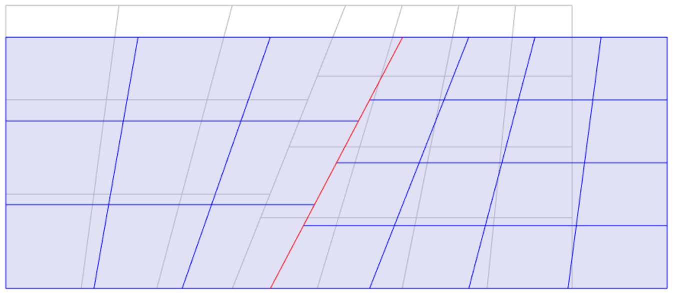
# Tied patch tests: errors on the order of machine precision for linear/quadratic elements in 2D/3D

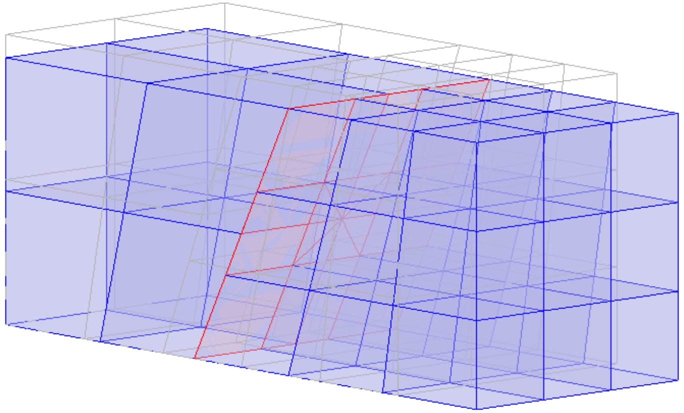
2D

3D

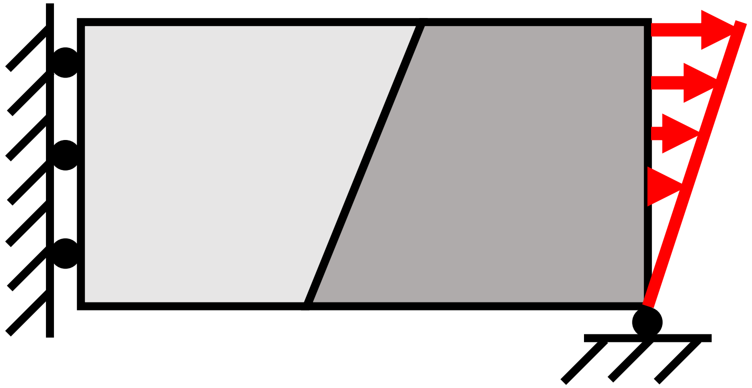
Linear tied patch test

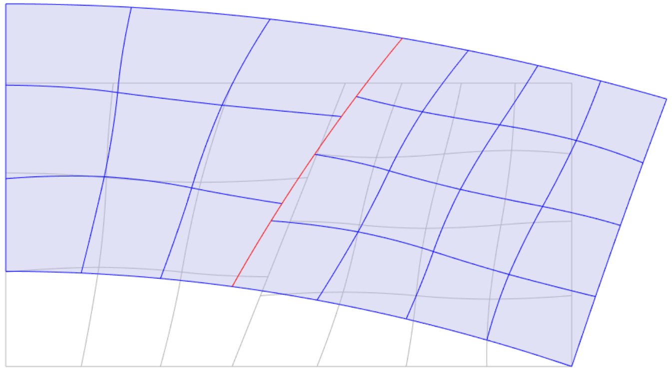


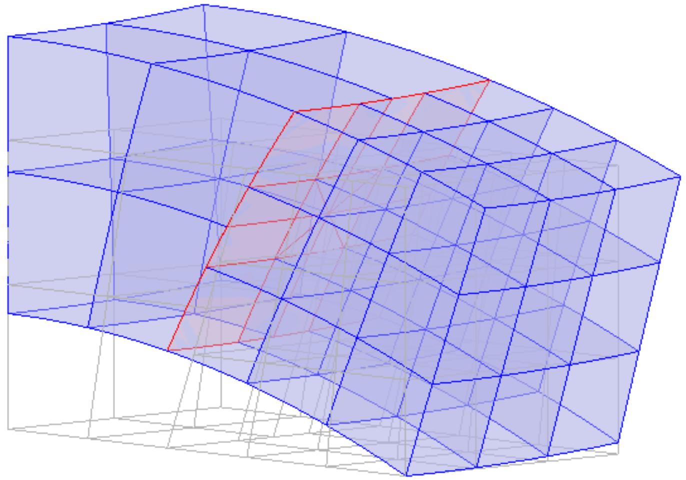




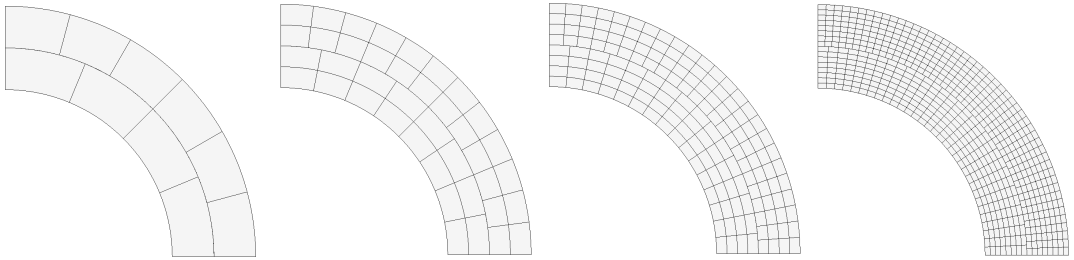
Quadratic tied patch test

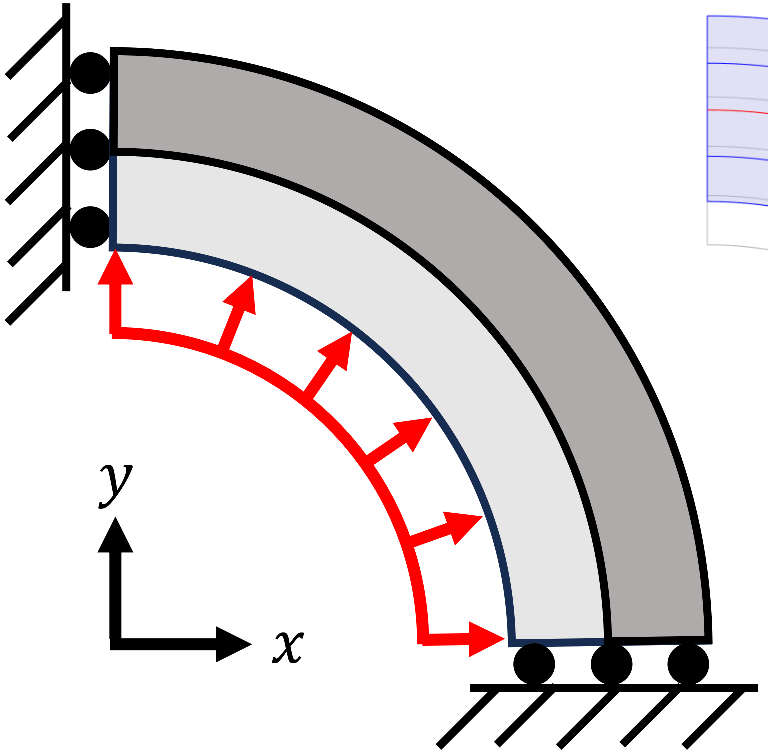


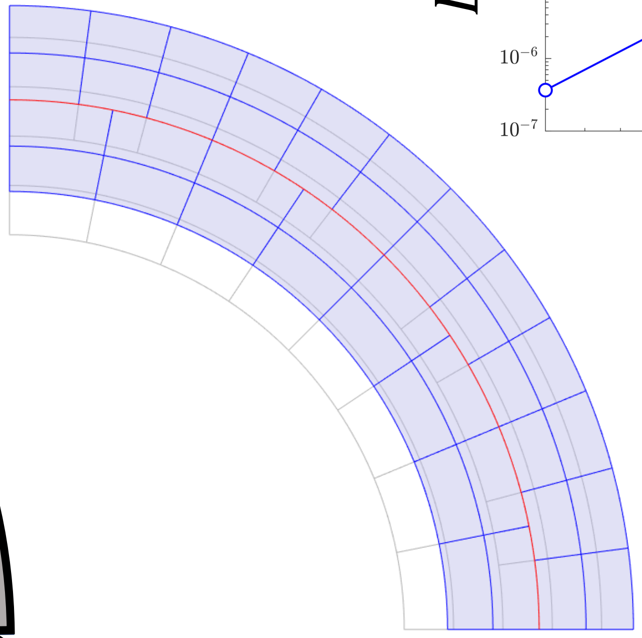


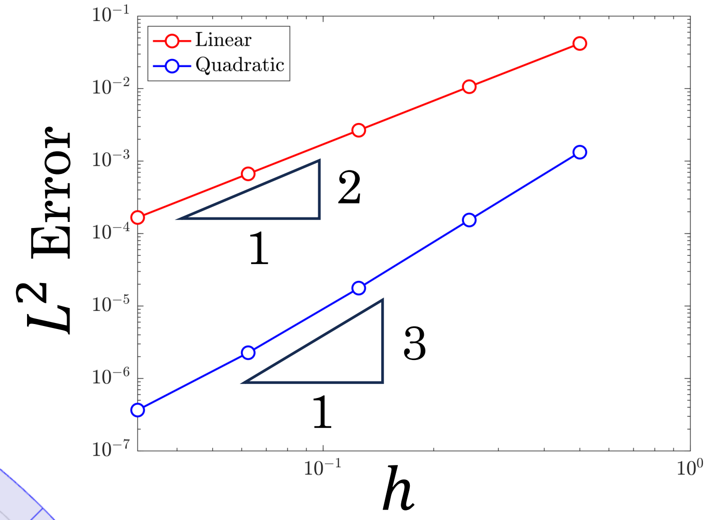


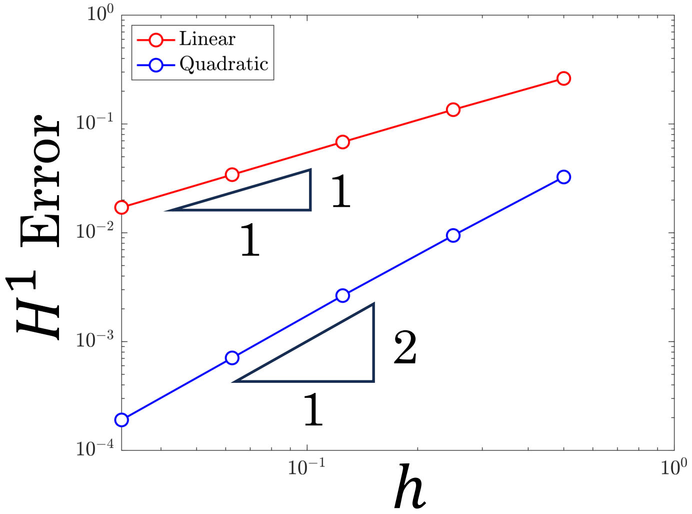
# Pressurized cylinder: $L^2$ and $H^1$ errors converge at expected rates using linear/quadratic elements

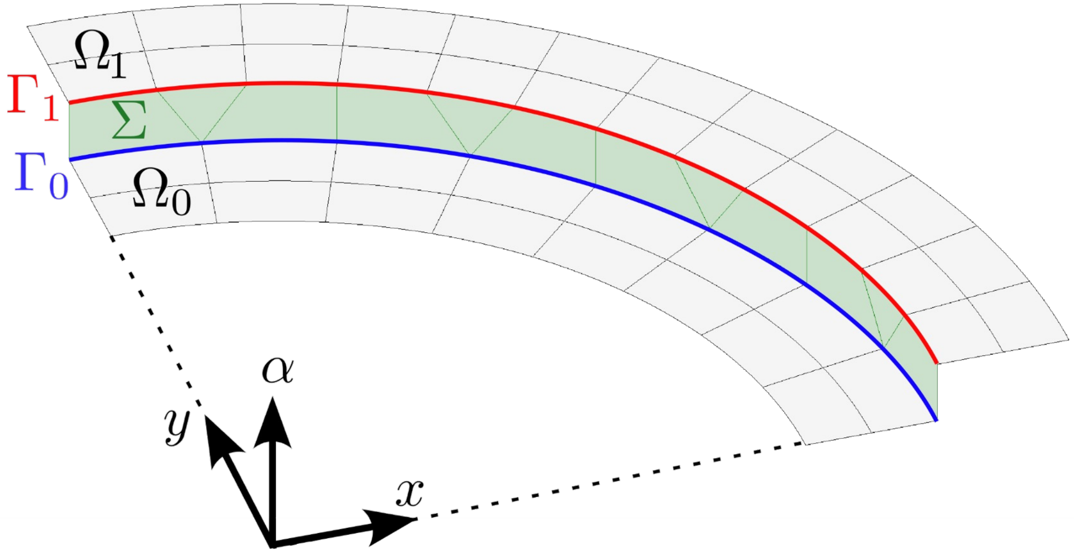




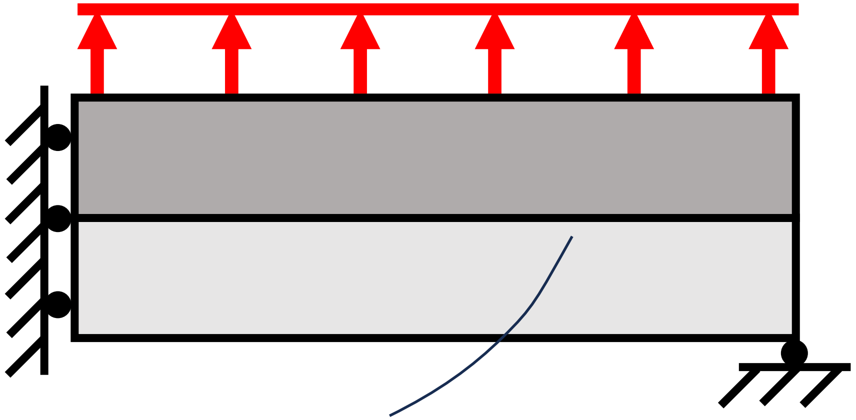





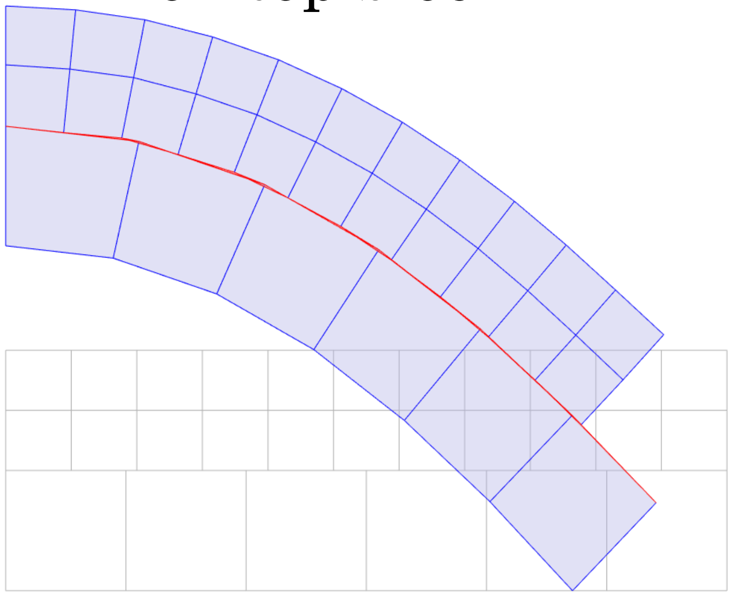



# Stacked cantilever beam: finite deformations with moderate sliding, showing locking-free behavior

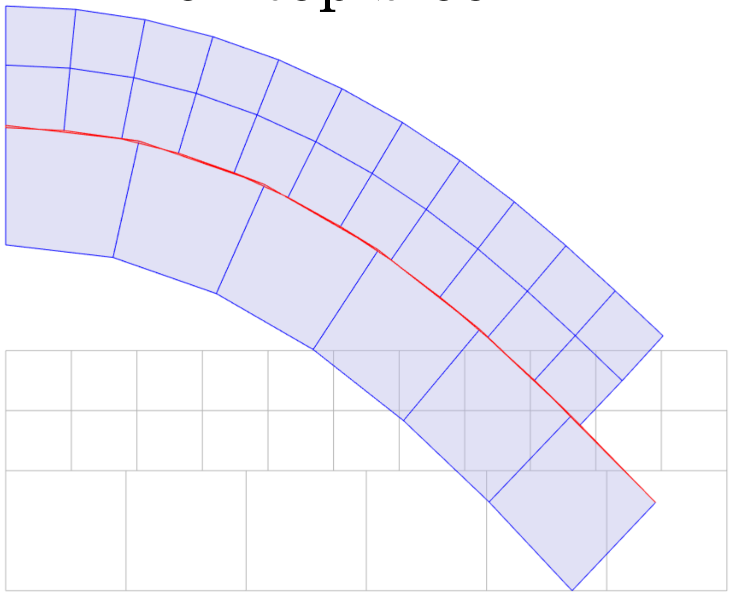



$\Gamma_1$ & multipliers defined on top block



$\Gamma_0$ defined on bottom block

$\Gamma_0$ defined on top block



$\Gamma_1$ & multipliers defined on bottom block

**Maximum displacement differs by 0.13%**

# The proposed method offers several advantages

1. Does not require the computation of geometric intersections or projections
   - Requires conformal meshing of the intermediate domain
2. Standard Gaussian quadrature is sufficient for satisfaction of patch tests and convergence
3. Natural and efficient extension to higher-order discretizations

# Ongoing and future work will seek to explore the following areas of continuing interest:

- Large sliding problems with separation and friction
- Stable symmetric dual-pass mortar formulations
- Surface-to-surface mesh solution remapping
- Locking-free penalty formulations
- Alternative discretization methods:
  - Isogeometric surfaces
  - Boundary element method
  - Polyhedral elements
  - Mesh-free methods
  - ALE methods

# Questions?